Spring 2014 **Math 601** Name: ______________________

Quiz 8

Question 1. (10 pts)

(a) Determine whether the function $f(z) = x^2 - y^2 + 2x + i(2xy + 2y)$ is analytic on $\mathbb{C}$.

Solution: $f(z) = x^2 - y^2 + 2x + i(2xy + 2y)$. So the real part is $u(x, y) = x^2 - y^2 + 2x$ and the imaginary part is $v(x, y) = 2xy + 2y$. We have

$$\frac{\partial u}{\partial x} = 2x + 2, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x + 2.$$

The Cauchy-Riemann equations are indeed satisfied. Therefore, f is analytic on $\mathbb{C}$.

(b) Specify the domain where

$$g(z) = \frac{z + 2\pi i}{z^2 + 4}$$

is analytic. Find the derivative of $g(z)$.

Solution: $g(z)$ is defined when $z^2 + 4 \neq 0$, that is, when $z \neq \pm 2i$. So the domain of g is

$$\mathbb{C} \backslash \{\pm 2i\}.$$

The derivative of g is

$$g'(z) = \frac{(z + 2\pi i)'(z^2 + 4) - (z + 2\pi i)(z^2 + 4)'}{(z^2 + 4)^2} = \frac{4 - 4\pi i z - z^2}{(z^2 + 4)^2}$$

Question 2. (10 pts)

Evaluate the following integrals.

(a)

$$\int_C (e^z + \cos z)dz$$

where C consists of two parts C_1 and C_2. C_1 is the line segment from the origin to $(1,1)$ and C_2 is the curve $y = x^2$ from $(1,1)$ to $(2,4)$.

Solution: Note that $F(z) = e^z + \sin z$ is the antiderivative of $(e^z + \cos z)$. Since the domain of $f(z) = e^z + \cos z$ is $\mathbb{C}$, hence simply-connected, and the curve C is clearly contained in $\mathbb{C}$. The two end points are

$$0 \quad \text{and} \quad (2+4i)$$

We know that

$$\int_C z^3 dz = F(2+4i) - F(0) = e^{2+4i} + \sin(2+4i) - 1$$

(b)

$$\int_C \frac{e^z}{(z+10)(z-5i)^2} dz$$

where C is the circle $\{z \in \mathbb{C} : |z-1| = 1\}$ oriented counterclockwise.

Solution: Notice that the function $f(z) = \frac{e^z}{(z+10)(z-5i)^2}$ is analytic on the region $\Omega = \mathbb{C} \backslash \{-10, 5i\}$. The circle $C = \{z \in \mathbb{C} : |z-1| = 1\}$ is contained in Ω with the inside of C lying in Ω. So we can apply Cauchy's Theorem, which implies that

$$\int_C \frac{e^z}{(z+10)(z-5i)^2} dz = 0$$